

Indian Statistical Institute, Bangalore Centre.
Mid-Semester Exam : Markov Chains (M2)

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Date : September 14, 2016.

Max. points : 40.

Time Limit : 3 hours.

Answer any two questions.

Give complete proofs. Please cite/quote appropriate results from class or assignments properly.

1. Let $f_{i,j} = \mathbb{P}_i(T_j < \infty)$, $r_{i,j} = \mathbb{E}_i(N_j)$ where T_j is the return time to j and N_j is the number of visits (including X_0) to j . Show that the following hold :
 - For $m \geq 1$, $\mathbb{P}_j(N_j = m) = f_{j,j}^{m-1}(1 - f_{j,j})$ **(5)**
 - For $i \neq j$,
$$\mathbb{P}_i(N_j = 0) = 1 - f_{i,j} ; \mathbb{P}_i(N_j = m) = f_{i,j} f_{j,j}^{m-1}(1 - f_{j,j}), m \geq 1. \text{ **(10)}**$$
 - $r_{j,j} = (1 - f_{j,j})^{-1}$ and for $i \neq j$, $r_{i,j} = f_{i,j} r_{j,j}$. **(5)**
2. (Wright-Fisher Model of Genetic drift): A population of m organisms carries two genes each and so $2m$ genes in total. Each gene is one of the two types - a or A. At each generation, the population reproduces by sampling $2m$ genes at random with replacement from the current generation of $2m$ genes. In other word, every generation has $2m$ genes and each gene independently chooses its type (a or A) uniformly at random from the genes of the previous generation. Let X_n be the number of genes of type a in the n th generation. Analyse the following properties of the chain.
 - (a) Show that X_n is an HMC and calculate its transition matrix. **(5)**
 - (b) What are the communication classes ? What is the period of the chain ? Is the chain reversible ? **(5)**
 - (c) What are the transient, null recurrent and positive recurrent states ? **(5)**

- (d) What are the possible stationary distributions for the chain ? **(5)**
3. (Renewal Reward process): Let X_n denote the pocket money of Shyam at the end of day n . Everyday Shyam spends Rs 1 and if at the beginning of the day n he has no money his father gives him Rs Y_n where $Y_n, n \geq 1$ are i.i.d random variables with density function given by $f_j, j = 1, 2, \dots$ i.e., $P(Y_n = j) = f_j$. His father gives him money only if he has no money and not otherwise. Analyse the following properties of the chain.
- (a) Show that X_n is an HMC and calculate its transition matrix. What are the communication classes ? **(5)**
- (b) What are the transient, null recurrent and positive recurrent states ? Calculate $E_0(T_0)$. **(5)**
- (c) Is the chain reversible ? Does a stationary distribution exist and if so, is it unique ? **(5)**
- (d) If Z_n is the total pocket money received by Shyam as on day n , then calculate the limit Z_n/n (if it exists). **(5)**

The answers may depend on suitable assumptions on f_j 's and in such a case specify under what conditions on f_j 's, which properties hold. Try to cover all possible cases of f_j 's as possible.